Companions, Codensity and Causality

Damien Pous, Jurriaan Rot

Companions, Codensity and Causality*

Damien Pous[1] and Jurriaan Rot[2]

[1] Univ Lyon, CNRS, ENS de Lyon, UCB Lyon 1, LIP – France
[2] Radboud University, Nijmegen, and CWI, Amsterdam – The Netherlands

Abstract. In the context of abstract coinduction in complete lattices, the notion of compatible function makes it possible to introduce enhancements of the coinduction proof principle. The largest compatible function, called the companion, subsumes most enhancements and has been proved to enjoy many good properties. Here we move to universal coalgebra, where the corresponding notion is that of a *final* distributive law. We show that when it exists the final distributive law is a monad, and that it coincides with the codensity monad of the final sequence of the given functor. On sets, we moreover characterise this codensity monad using a new abstract notion of causality. In particular, we recover the fact that on streams, the functions definable by a distributive law or GSOS specification are precisely the causal functions. Going back to enhancements of the coinductive proof principle, we finally obtain that any causal function gives rise to a valid up-to-context technique.

1 Introduction

Coinduction has been widely studied since Milner's work on CCS [26]. In concurrency theory, it is usually exploited to define behavioural equivalences or preorders on processes and to obtain powerful proof principles. Coinduction can also be used for programming languages, to define and manipulate infinite data-structures like streams or potentially infinite trees. For instance, streams can be defined using systems of differential equations [36]. In particular, pointwise addition of two streams x, y can be defined by the following equations, where x_0 and x' respectively denote the head and the tail of the stream x.

$$\begin{aligned}(x \oplus y)_0 &= x_0 + y_0 \\ (x \oplus y)' &= x' \oplus y'\end{aligned} \tag{1}$$

Coinduction as a proof principle for concurrent systems can nicely be presented at the abstract level of complete lattices [30,32]: bisimilarity is the greatest fixpoint of a monotone function on the complete lattice of binary relations. In contrast, coinduction as a tool to manipulate infinite data-structures requires

* The research leading to these results has received funding from the European Research Council (FP7/2007-2013, grant agreement nr. 320571 ; and H2020, grant agreement nr. 678157); as well as from the LABEX MILYON (ANR-10-LABX-0070, ANR-11-IDEX-0007) and the project PACE (ANR-12IS02001).

one more step to be presented abstractly: moving to universal coalgebra [15]. For instance, streams are the carrier of the final coalgebra of an endofunctor on Set, and simple systems of differential equations are just plain coalgebras.

In both cases one frequently needs enhancements of the coinduction principle [37,38]. Indeed, rather than working with plain bisimulations, which can be rather large, one often uses "bisimulations up-to", which are not proper bisimulations but are nevertheless contained in bisimilarity [27,2,1,10,16,24,39]. The situation with infinite data-structures is similar. For instance, defining the shuffle product on streams is typically done using equations of the following shape,

$$\begin{aligned}(x \otimes y)_0 &= x_0 \times y_0 \\ (x \otimes y)' &= x \otimes y' \;\oplus\; x' \otimes y\end{aligned} \tag{2}$$

which fall out of the scope of plain coinduction due to the call to pointwise addition [36,12].

Enhancements of the bisimulation proof method have been introduced by Milner from the beginning [26], and further studied by Sangiorgi [37,38] and then by the first author [30,32]. Let us recall the standard formulation of coinduction in complete lattices: by Knaster-Tarski's theorem [19,41], any monotone function b on a complete lattice admits a greatest fixpoint νb that satisfies the following *coinduction principle*:

$$\frac{x \leq y \leq b(y)}{x \leq \nu b}\ \textsc{coinduction} \tag{3}$$

In words, to prove that some point x is below the greatest fixpoint, it suffices to exhibit a point y above x which is an *invariant*, i.e., a post-fixpoint of b. Enhancements, or up-to techniques, make it possible to alleviate the second requirement: instead of working with post-fixpoints of b, one might use post-fixpoints of $b \circ f$, for carefully chosen functions f:

$$\frac{x \leq y \leq b(f(y))}{x \leq \nu b}\ \textsc{coinduction up to}\ f \tag{4}$$

Taking inspiration Hur et al.' work [13], the first author recently proposed to systematically use for f the largest *compatible* function [31], i.e., the largest function t such that $t \circ b \leq b \circ t$. Such a function always exists and is called the *companion*. It enjoys many good properties, the most important one possibly being that it is a closure operator: $t \circ t = t$. Parrow and Weber also characterised it extensionally in terms of the final sequence of the function b [29,31]:

$$t : x \mapsto \bigwedge_{x \leq b_\alpha} b_\alpha \qquad \text{where} \begin{cases} b_\lambda \triangleq \bigwedge_{\alpha<\lambda} b_\alpha & \text{for limit ordinals} \\ b_{\alpha+1} \triangleq b(b_\alpha) & \text{for successor ordinals} \end{cases} \tag{5}$$

In the present paper, we give a categorical account of these ideas, generalising them from complete lattices to universal coalgebra, in order to encompass important instances of coinduction such as solving systems of equations on infinite data-structures.

Let us first be more precise about our example on streams. We consider there the Set functor $BX = \mathbb{R} \times X$, whose final coalgebra is the set $\mathbb{R}^\omega$ of streams over the reals. This means that any B-coalgebra (X, f) defines a function from X to streams. Take for instance the following coalgebra over the two-elements set $2 = \{0, 1\}$: $0 \mapsto (0.3,\ 1)$, $1 \mapsto (0.7,\ 0)$. This coalgebra can be seen as a system of two equations, whose unique solution is a function from 2 to $\mathbb{R}^\omega$, i.e, two streams, where the first has value 0.3 at all even positions and 0.7 at all odd positions.

In a similar manner, one can define binary operations on streams by considering coalgebras whose carrier consists of pairs of streams. For instance, the previous system of equations characterising pointwise addition (1) is faithfully represented by the following coalgebra:

$$\begin{aligned}(\mathbb{R}^\omega)^2 &\to B((\mathbb{R}^\omega)^2)\\ (x, y) &\mapsto (x_0 + y_0,\ \ (x', y'))\end{aligned}$$

Unfortunately, as explained above, systems of equations defining operations like shuffle product (2) cannot be represented easily in this way: we would need to call pointwise addition on streams that are not yet fully defined.

To this end, one can weaken the requirement of a B-coalgebra to that of a BF-coalgebra, when there exists a distributive law $\lambda : FB \Rightarrow BF$ of a monad F over B [5,12]. The proof relies on the so-called generalised powerset construction [40], and this precisely amounts to using an up-to technique. Such a use of distributive laws is actually rather standard in operational semantics [42,5,17]; they properly generalise the notion of compatible function. In order to follow [31], we thus focus on the largest distributive law.

Our first contribution consists in showing that if a functor B admits a final distributive law (called the companion), then 1) this distributive law is that of a monad T over B, and 2) any BT-coalgebra has a unique morphism to the final B-coalgebra, representing a solution to the system of equations modeled by the coalgebra (Section 3). In complete lattices, this corresponds to the facts that the companion is a closure operator and that it can be used as an up-to technique.

Then we move to conditions under which the companion exists. We start from the *final sequence* of the functor B, which is commonly used to obtain the existence of a final coalgebra [3,4], and we show that the companion actually coincides with the *codensity monad* of this sequence, provided that this codensity monad exists and is preserved by B (Theorem 5.1). Those conditions are satisfied by all polynomial functors. This link with the final sequence of the functor makes it possible to recover Parrow and Weber's characterisation (Equation (5)).

We can go even further for ω-continuous endofunctors on Set: the codensity monad of the final sequence can be characterised in terms of a new abstract notion of *causal algebra* (Definition 6.1). On streams, this notion coincides with the standard notion of causality [12]: causal algebras (on streams) correspond to operations such that the n-th value of the result only depends on the n-th first values of the arguments. For instance, pointwise addition and shuffle product are causal algebras for the functor $SX = X^2$.

These two characterisations of the companion in terms of the codensity monad and in terms of causal algebras are the key theorems of the present paper. We study some of their consequences in Section 7.

First, given a causal algebra for a functor F, we get that any system of equations represented as a BF-coalgebra admits a unique solution. Such a technique makes it possible to define shuffle product in a streamlined way, without using distributive laws: using pointwise stream addition as a causal S-algebra, Equations (2) can be represented by the following BS-coalgebra:

$$\begin{aligned} &(\mathbb{R}^\omega)^2 \to BS((\mathbb{R}^\omega)^2) \\ &(x, y) \mapsto (x_0 \times y_0,\ \ ((x, y')\,,\,(x', y))) \end{aligned}$$

(Intuitively, the inner pairs (x, y') and (x', y) correspond to the corecursive calls, and thus to the shuffle products $x \otimes y'$ and $x' \otimes y$; in contrast, the intermediate pair $((x, y')\,,\,(x', y))$ corresponds to a call to the causal algebra on S, i.e., in this case, pointwise addition.) In the very same way, with the functor $BX = 2 \times X^A$ for deterministic automata, we immediately obtain the semantics of non-deterministic automata and context-free grammars using simple causal algebras on formal languages (Examples 7.1 and 7.2).

Second, we obtain that algebras on the final coalgebra are causal if and only if they can be defined by a distributive law. Similar results were known to hold for streams [12] and languages [34]. Our characterisation is more abstract and less syntactic; the precise relationship between those results remains to be studied.

Third, we can combine our results with some recent work [6] where we rely on (bi)fibrations to lift distributive laws on systems (e.g., automata, LTSs) to obtain up-to techniques for coinductive predicates or relations on those systems (e.g., language equivalence, bisimilarity, divergence). Doing so, we obtain that every causal algebra gives rise to a valid up-to context technique (Section 7.3). For instance, bisimulation up to pointwise additions and shuffle products is a valid technique for proving stream equalities coinductively.

We conclude with an expressivity result (Section 8): while abstract GSOS specifications [42] seem more expressive than plain distributive laws, we show that this is actually not the case: any algebra obtained from an abstract GSOS specification can actually be defined from a plain distributive law.

Acknowledgments. We are grateful to Henning Basold, Filippo Bonchi, Bart Jacobs, Joshua Moerman, Daniela Petrişan, and Jan Rutten for valuable discussions and comments.

2 Preliminaries

A *coalgebra* for a functor $B\colon \mathcal{C} \to \mathcal{C}$ is a pair (X, f) where X is an object in $\mathcal{C}$ and $f\colon X \to BX$ a morphism. A coalgebra homomorphism from (X, f) to (Y, g) is a $\mathcal{C}$-morphism $h\colon X \to Y$ such that $g \circ h = Fh \circ f$. A coalgebra (Z, ζ) is called *final* if it is final in the category of coalgebras, i.e., for every coalgebra (X, f) there exists a unique coalgebra morphism from (X, f) to (Z, ζ).

An *algebra* for a functor $F : \mathcal{D} \to \mathcal{D}$ is defined dually to a coalgebra, i.e., it is a pair (X, a) where $a : FX \to X$, and an algebra morphism from (X, a) to (Y, b) is a morphism $h : X \to Y$ such that $h \circ a = b \circ Fh$.

A *monad* is a triple (T, η, μ) where $T : \mathcal{C} \to \mathcal{C}$ is a functor, and $\eta : \mathsf{Id} \Rightarrow T$ and $\mu : TT \Rightarrow T$ are natural transformations called *unit* and *multiplication* respectively, such that $\mu \circ T\eta = \mathsf{id} = \mu \circ \eta_T$ and $\mu \circ \mu_T = \mu \circ T\mu$.

Distributive laws. A *distributive law* of a functor $F : \mathcal{C} \to \mathcal{C}$ over a functor $B : \mathcal{C} \to \mathcal{C}$ is a natural transformation $\lambda : FB \Rightarrow BF$. If B has a final coalgebra (Z, ζ), then such a λ induces a unique algebra α making the following commute.

$$\begin{tikzcd} FZ \arrow[r, "F\zeta"] \arrow[d, "\alpha"'] & FBZ \arrow[r, "\lambda_Z"] & BFZ \arrow[d, "B\alpha"] \\ Z \arrow[rr, "\zeta"'] & & BZ \end{tikzcd}$$

We call α the *algebra induced by* λ (on the final coalgebra).

Let (T, η, μ) be a monad. A distributive law of (T, η, μ) over B is a natural transformation $\lambda : TB \Rightarrow BT$ such that $B\eta = \lambda \circ \eta_B$ and $\lambda \circ \mu_B = B\mu \circ \lambda_T \circ T\lambda$.

Final sequence. Let $B : \mathcal{C} \to \mathcal{C}$ be an endofunctor on a complete category $\mathcal{C}$. The *final sequence* is the unique ordinal-indexed sequence defined by $B_0 = 1$ (the final object of $\mathcal{C}$), $B_{i+1} = BB_i$ and $B_j = \lim_{i<j} B_i$ for a limit ordinal j, with connecting morphisms $B_{j,i} : B_j \to B_i$ for all $i \leq j$, satisfying $B_{i,i} = \mathsf{id}$, $B_{j+1,i+1} = BB_{j,i}$ and if j is a limit ordinal then $(B_{j,i})_{i<j}$ is a limit cone.

The final sequence is a standard tool for constructing final coalgebras: if there exists an ordinal k such that $B_{k+1,k}$ is an isomorphism, then $B^{-1}_{k+1,k} : B_k \to BB_k$ is a final B-coalgebra [4, Theorem 1.3] (and dually for initial algebras [3]). In the sequel, we shall sometimes present it as a functor $\bar{B} : \mathsf{Ord}^{\mathsf{op}} \to \mathcal{C}$, given by $\bar{B}(i) = B_i$ and $\bar{B}(j, i) = B_{j,i}$.

Example 2.1. Consider the functor $B : \mathsf{Set} \to \mathsf{Set}$ given by $BX = A \times X$, whose coalgebras are stream systems. Then $B_0 = 1$ and $B_{i+1} = A \times B_i$ for $0 < i < \omega$. Hence, for $i < \omega$, B_i is the set of all finite lists over A of length i. The limit B_ω consists of the set of all streams over A. For each i, j with $i \leq j$, the connecting map $B_{j,i}$ maps a stream (if $j = \omega$) or a list (if $j < \omega$) to the prefix of length i. The set B_ω of streams is a final B-coalgebra.

Example 2.2. For the Set functor $BX = 2 \times X^A$ whose coalgebras are deterministic automata over A, B_i is (isomorphic to) the set of languages of words over A with length below i. In particular, $B_\omega = \mathcal{P}(A^*)$ is the set of all languages, and it is a final B-coalgebra.

A functor $B : \mathcal{C} \to \mathcal{C}$ is called *(ω)-continuous* if it preserves limits of ω^{op}-chains. For such a functor, B_ω is the carrier of a final B-coalgebra. The functors of stream systems and automata in the above examples are both ω-continuous.

3 Properties of the companion

Definition 3.1. *Let $B \colon \mathcal{C} \to \mathcal{C}$ be a functor. The category* $\mathsf{DL}(B)$ *of distributive laws is defined as follows. An object is a pair (F, λ) where $F \colon \mathcal{C} \to \mathcal{C}$ is a functor and $\lambda \colon FB \Rightarrow BF$ is a natural transformation. A morphism from (F, λ) to (G, ρ) is a natural transformation $\kappa \colon F \Rightarrow G$ s.t. $\rho \circ \kappa_B = B\kappa \circ \lambda$. The* companion *of B is the final object of* $\mathsf{DL}(B)$*, if it exists.*

$$\begin{tikzcd} FB \arrow[r, Rightarrow, "\kappa B"] \arrow[d, Rightarrow, "\lambda"'] & GB \arrow[d, Rightarrow, "\rho"] \\ BF \arrow[r, Rightarrow, "B\kappa"'] & BG \end{tikzcd}$$

Morphisms in $\mathsf{DL}(B)$ are a special case of *morphisms of distributive laws*, see [33,43,22,18]. In the remainder of this section, we assume that the companion of B exists, and we denote it by (T, τ). We first prove that it is a monad.

Theorem 3.1. *There are unique $\eta \colon \mathsf{Id} \Rightarrow T$ and $\mu \colon TT \Rightarrow T$ such that (T, η, μ) is a monad and $\tau \colon TB \Rightarrow BT$ is a distributive law of this monad over B.*

Proof. Define η and μ as the unique morphisms from id_B and $\tau_T \circ T\tau$ respectively to the companion:

$$\begin{tikzcd} B \arrow[r, equal] \arrow[d, Rightarrow, "\eta B"'] & B \arrow[d, Rightarrow, "B\eta"] \\ TB \arrow[r, Rightarrow, "\tau"] & BT \end{tikzcd} \qquad \begin{tikzcd} TTB \arrow[r, Rightarrow, "T\tau"] \arrow[d, Rightarrow, "\mu B"'] & TBT \arrow[r, Rightarrow, "\tau T"] & BTT \arrow[d, Rightarrow, "B\mu"] \\ TB \arrow[rr, Rightarrow, "\tau"] & & BT \end{tikzcd}$$

By definition, they satisfy the required axioms for τ to be a distributive law of monad over functor. The proof that (T, η, μ) is indeed a monad is routine, using finality of (T, τ), see the appendix. □

A distributive law λ of a monad over a functor allows one to strengthen the coinduction principle obtained by finality, as observed in [5] (specifically its Corollary 4.3.6), where it is called *λ-coiteration.* This principle allows one to solve (co)recursive equations, see, e.g., loc. cit. and [14,25]. Since the companion is a distributive law of a monad (Theorem 3.1) we obtain the following.

Corollary 3.1. *Let (Z, ζ) be a final B-coalgebra. For every morphism $f \colon X \to BTX$ there is a unique morphism $f^\dagger \colon X \to Z$ such that the following commutes:*

$$\begin{tikzcd} X \arrow[rr, "f^\dagger"] \arrow[d, "f"'] & & Z \arrow[d, "\zeta"] \\ BTX \arrow[r, "BTf^\dagger"'] & BTZ \arrow[r, "B\alpha"'] & BZ \end{tikzcd}$$

where α is the algebra induced by the distributive law τ of the companion.

Instantiated to the complete lattice case, this is a soundness result: any invariant up to the companion (a post-fixpoint of $b \circ t$) is below the greatest fixpoint (νb).

Now assume that $\mathcal{C}$ has an initial object 0. One can define the final coalgebra and the algebra induced by the companion explicitly:

Theorem 3.2. *The B-coalgebra $(T0, \tau_0 \circ T!_{B0})$ is final, and the algebra induced on it by the companion is given by μ_0.*

More generally, the algebra induced by any distributive law factors through the algebra μ_0 induced by the companion.

Proposition 3.1. *Let (T, η, μ) be the monad on the companion (Theorem 3.1). Let $\lambda\colon FB \Rightarrow BF$ be a distributive law, and $\alpha\colon FT0 \Rightarrow T0$ the algebra on the final coalgebra induced by it. Let $\bar{\lambda}\colon F \Rightarrow T$ be the unique natural transformation induced by finality of the companion. Then $\alpha = \mu_0 \circ \bar{\lambda}_{T0}$.*

4 The codensity monad

The notion of *codensity monad* is a special instance of a right Kan extension, which plays a central role in the following sections. We briefly define them here; see [20,28,21] for a comprehensive study.

$$\begin{tikzcd} HF \arrow[rr, Rightarrow, "\kappa F"] \arrow[dr, Rightarrow, "\alpha"'] & & IF \arrow[dl, Rightarrow, "\beta"] \\ & G & \end{tikzcd}$$

Given $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{C} \to \mathcal{E}$ two functors. Define the category $\mathcal{K}(F, G)$ whose objects are pairs (H, α) of a functor $H\colon \mathcal{D} \to \mathcal{E}$ and a natural transformation $\alpha\colon HF \Rightarrow G$. A morphism from (H, α) to (I, β) is a natural transformation $\kappa\colon H \Rightarrow I$ such that $\beta \circ \kappa_F = \alpha$.

The *right Kan extension* of G along F is a final object $(\mathsf{Ran}_F G, \epsilon)$ in $\mathcal{K}(F, G)$; the natural transformation $\epsilon\colon (\mathsf{Ran}_F G)F \Rightarrow G$ is called its *counit*. A functor $K\colon \mathcal{E} \to \mathcal{F}$ is said to *preserve* $\mathsf{Ran}_F G$ if $K \circ \mathsf{Ran}_F G$ is a right Kan extension of KG along F, with counit $K\epsilon\colon K(\mathsf{Ran}_F G)F \Rightarrow KG$.

$$\begin{tikzcd} HF \arrow[rr, Rightarrow, "\hat{\alpha}F"] \arrow[dr, Rightarrow, "\alpha"'] & & \mathsf{C}_F F \arrow[dl, Rightarrow, "\epsilon"] \\ & F & \end{tikzcd}$$

The codensity monad is a special case, with $F = G$. Explicitly, the *codensity monad* of a functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a functor $\mathsf{C}_F\colon \mathcal{D} \to \mathcal{D}$ and a natural transformation $\epsilon\colon \mathsf{C}_F F \Rightarrow F$ s.t. for every functor $H\colon \mathcal{D} \to \mathcal{D}$ and natural transformation $\alpha\colon HF \Rightarrow F$ there is a unique $\hat{\alpha}\colon H \Rightarrow \mathsf{C}_F$ s.t. $\epsilon \circ \hat{\alpha}_F = \alpha$.

As the name suggests, C_F is a monad: the unit η and the multiplication μ are the unique natural transformations such that $\epsilon \circ \eta_F = \mathsf{id}$ and $\epsilon \circ \mu_F = \epsilon \circ \mathsf{C}_F \epsilon$. In the sequel we will abbreviate the category $\mathcal{K}(F, F)$ as $\mathcal{K}(F)$.

Right Kan extensions can be computed pointwise as a limit, if sufficient limits exist. For an object X in $\mathcal{D}$, denote by $\Delta_X : \mathcal{C} \to \mathcal{D}$ the functor that maps every object to X. By Δ_X/F we denote the comma category, where an object is a pair (Y, f) consisting of an object Y in $\mathcal{C}$ and an arrow $f\colon X \to FY$ in $\mathcal{D}$, and an arrow from (Y, f) to (Z, g) is a map $h\colon Y \to Z$ in $\mathcal{C}$ such that $Fh \circ f = g$. There is a forgetful functor $(\Delta_X/F) \to \mathcal{C}$, which remains unnamed below.

Lemma 4.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{C} \to \mathcal{E}$ be functors. If, for every object X in $\mathcal{D}$, the limit $\lim\left((\Delta_X/F) \to \mathcal{C} \xrightarrow{G} \mathcal{D}\right)$ exists, then the right Kan extension $\mathsf{Ran}_F G$ exists, and is given on an object X by that limit.*

The codensity monad of a functor F is the right Kan extension of F along itself. Hence, Lemma 4.1 gives us a way of computing the codensity monad.

The hypotheses are met in particular if $\mathcal{C}$ is essentially small (equivalent to a category with a set of objects and a set of arrows) and $\mathcal{D}$ is locally small and complete. The latter conditions hold for $\mathcal{D} = \mathsf{Set}$. In that case, we have the following concrete presentation; see, e.g., [8, Section 2.5] for a proof.

Lemma 4.2. *Let $F\colon \mathcal{C} \to \mathsf{Set}$ be a functor, where $\mathcal{C}$ is essentially small. The codensity monad C_F is given by $\mathsf{C}_F(X) = \{\alpha\colon (F-)^X \Rightarrow F\}$ and, for $h\colon X \to Y$, $(\mathsf{C}_F(h)(\alpha))_A\colon (FA)^Y \to FA$ is given by $f \mapsto \alpha_A(f \circ h)$. The natural transformation $\epsilon\colon \mathsf{C}_F F \Rightarrow F$ is given by $\epsilon_X(\alpha\colon F^{FX} \Rightarrow F) = \alpha_X(\mathsf{id}_{FX})$.*

5 Constructing the companion by codensity

It is standard in the theory of coalgebras to compute the final coalgebra of a functor B as a limit of the final sequence $\bar{B}$, see Section 2. In this section, we focus on the codensity monad of the final sequence, and show that it yields—under certain conditions—the companion of B.

The codensity monad of $\bar{B}$ is final in the category of natural transformations of the form $F\bar{B} \Rightarrow \bar{B}$ (see Section 4), whereas the companion of B is final in the category of distributive laws over B. The following lemma is a first step towards connecting companion and codensity monad.

Lemma 5.1. *For every $\lambda\colon FB \Rightarrow BF$ there exists a unique $\alpha\colon F\bar{B} \Rightarrow \bar{B}$ such that for all $i \in \mathsf{Ord}$: $\alpha_{i+1} = B\alpha_i \circ \lambda_{B_i}$. Moreover, if $B_{k+1,k}$ is an isomorphism for some k, then α_k is the algebra induced by λ on the final coalgebra.*

We turn to the main result of this section: the codensity monad of $\bar{B}$ yields the companion of B, if B preserves this codensity monad. The latter condition, as well as the concrete form of the companion computed in this manner, becomes clearer when we instantiate this result to the case where $\mathcal{C}$ is a lattice (Section 5.1) and the case $\mathcal{C} = \mathsf{Set}$ (Section 6).

Theorem 5.1. *Let $\bar{B}\colon \mathsf{Ord}^{\mathsf{op}} \to \mathcal{C}$ be the final sequence of an endofunctor B. If the codensity monad $\mathsf{C}_{\bar{B}}$ exists and B preserves it (as a right Kan extension) then there is a distributive law τ of the codensity monad $(\mathsf{C}_{\bar{B}}, \eta, \mu)$ over B such that $(\mathsf{C}_{\bar{B}}, \tau)$ is the companion of B.*

Proof (Outline). The preservation assumption means that $(B\mathsf{C}_{\bar{B}}, B\epsilon)$ is a right Kan extension of $B\bar{B}$ along $\bar{B}$. The natural transformation τ is defined, using the universal property of $B\epsilon$, as the unique $\tau\colon \mathsf{C}_{\bar{B}}B \Rightarrow B\mathsf{C}_{\bar{B}}$ such that $B\epsilon_i \circ \tau_{B_i} = \epsilon_{i+1}\colon \mathsf{C}_{\bar{B}}BB_i \Rightarrow BB_i$ for all i. See the appendix for a full proof. $\square$

The following result characterises the algebra induced on the final coalgebra by the distributive law of the companion, in terms of the counit ϵ of the codensity monad of $\bar{B}$. This plays an important role for the case $\mathcal{C} = \mathsf{Set}$ (Section 7).

Proposition 5.1. *Suppose B is a functor satisfying the hypotheses of Theorem 5.1. Let $(\mathsf{C}_{\bar{B}}, \epsilon)$ be the codensity monad of $\bar{B}$, with distributive law τ and monad structure $(\mathsf{C}_{\bar{B}}, \eta, \mu)$. If $B_{k+1,k}$ is an isomorphism for some k, then*

1. *$\epsilon_k\colon \mathsf{C}_{\bar{B}}B_k \to B_k$ is the algebra induced by τ on the final coalgebra;*
2. *if $\mathcal{C}$ has an initial object 0 then ϵ_k is isomorphic to μ_0.*

5.1 Codensity and the companion of a monotone function

Throughout this section, let $b\colon L \to L$ be a monotone function on a complete lattice. By Theorem 5.1, the companion of a monotone function b (viewed as a functor on a poset category) is given by the right Kan extension of the final sequence $\bar{b}\colon \mathsf{Ord}^{\mathsf{op}} \to L$ along itself. Using Lemma 4.1, we obtain the characterisation of the companion given in the Introduction (5).

Theorem 5.2. *The companion t of b is given by*

$$t : x \mapsto \bigwedge_{x \le b_i} b_i$$

Proof. By Lemma 4.1, the codensity monad $\mathsf{C}_{\bar{b}}$ can be computed by

$$\mathsf{C}_{\bar{b}}(x) = (\mathsf{Ran}_{\bar{b}}\bar{b})(x) = \bigwedge_{x \le b_i} b_i \ ,$$

a limit that exists since L is a complete lattice. We apply Theorem 5.1 to show that $\mathsf{C}_{\bar{b}}$ is the companion of b. The preservation condition of the theorem amounts to the equality $b \circ \mathsf{Ran}_{\bar{b}}\bar{b} = \mathsf{Ran}_{\bar{b}}(b \circ \bar{b})$ which, by Lemma 4.1, in turn amounts to

$$b(\bigwedge_{x \le b_i} b_i) = \bigwedge_{x \le b_i} b(b_i)$$

for all $x \in L$. The sequence $(b_i)_{i \in \mathsf{Ord}}$ is decreasing and stagnates at some ordinal ϵ; therefore, the two intersections collapse into their last terms, say b_δ and $b(b_\delta)$ (with δ the greatest ordinal such that $x \not\le b_{\delta+1}$, or ϵ if such an ordinal does not exist). The equality follows. □

In fact, the category $\mathcal{K}(b)$ defined in Section 4 instantiates to the following: an object is a monotone function $f\colon L \to L$ such that $f(b_i) \le b_i$ for all $i \in \mathsf{Ord}$, and an arrow from f to g exists iff $f \le g$. The companion t is final in this category. This yields the following characterisation of functions below the companion.

Proposition 5.2. *Let t be the companion of b. For any monotone function f we have $f \le t$ iff $\forall i \in \mathsf{Ord} : f(b_i) \le b_i$.*

A key intuition about up-to techniques is that they should at least preserve the greatest fixpoint (i.e., up-to context is valid only when bisimilarity is a congruence). It is however well-known that this is not a sufficient condition [37,38]. The above proposition gives a stronger and better intuition: a technique should preserve all approximations of the greatest fixpoint (the elements of the final sequence) to be below the companion, and thus sound.

This intuition on complete lattices leads us to the abstract notion of causality we introduce in the following section.

6 Causality by codensity

We focus on the codensity monad of the final sequence of an ω-continuous Set endofunctor B. For such a functor, B_ω is the carrier of a final coalgebra and Lemma 4.2 provides us with a description of the codensity monad in terms of natural transformations of the form $(\bar{B}-)^X \Rightarrow \bar{B}$. We show that such natural transformations correspond to a new abstract notion which we call *causal algebras*. Based on this correspondence and Theorem 5.1, we will get a concrete understanding of the companion of B in Section 7.

Definition 6.1. *Let $B, F \colon \mathsf{Set} \to \mathsf{Set}$ be functors. An algebra $\alpha \colon FB_\omega \to B_\omega$ is called* (ω)-causal *if for every set X, functions $f, g \colon X \to B_\omega$ and $i < \omega$:*

$$\begin{tikzcd}[row sep=small] & B_\omega \arrow[dr, "B_{\omega,i}"] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & & B_i \\ & B_\omega \arrow[ur, "B_{\omega,i}"'] & \end{tikzcd} \quad \textit{implies} \quad \begin{tikzcd}[row sep=small] & FB_\omega \arrow[r, "\alpha"] & B_\omega \arrow[dr, "B_{\omega,i}"] & \\ FX \arrow[ur, "Ff"] \arrow[dr, "Fg"'] & & & B_i \\ & FB_\omega \arrow[r, "\alpha"'] & B_\omega \arrow[ur, "B_{\omega,i}"'] & \end{tikzcd}$$

Causal algebras form a category $\mathsf{causal}(B)$*: an object is a pair $(F, \alpha \colon FB_\omega \to B_\omega)$ where α is causal, and a morphism from (F, α) to (G, β) is a natural transformation $\kappa \colon F \Rightarrow G$ such that $\beta \circ \kappa_{B_\omega} = \alpha$.*

An (ω)-causal function *on $|V|$ arguments is a causal algebra for the functor $(-)^V$. Equivalently, $\alpha \colon (B_\omega)^V \to B_\omega$ is causal iff for every $h, k \in (B_\omega)^V$ and every $i < \omega$: if $B_{\omega,i} \circ h = B_{\omega,i} \circ k$ then $B_{\omega,i} \circ \alpha(h) = B_{\omega,i} \circ \alpha(k)$.*

Example 6.1. Recall from Example 2.1 that, for the functor $BX = A \times X$, B_i is the set of lists of length i, and in particular B_ω is the set of streams over A. We focus first on causal *functions*. To this end, for $\sigma, \tau \in B_\omega$, we write $\sigma \equiv_i \tau$ if σ and τ are equal up to i, i.e., $\sigma(k) = \tau(k)$ for all $k < i$. It is easy to verify that a function of the form $\alpha \colon (B_\omega)^n \to B_\omega$ is causal iff for all $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n$ and all $i < \omega$: if $\sigma_j \equiv_i \tau_j$ for all $j \leq n$ then $\alpha(\sigma_1, \ldots, \sigma_n) \equiv_i \alpha(\tau_1, \ldots, \tau_n)$.

For instance, taking $n = 2$, $\mathsf{alt}(\sigma, \tau) = (\sigma(0), \tau(1), \sigma(2), \tau(3), \ldots)$ is causal, whereas $\mathsf{even}(\sigma) = (\sigma(0), \sigma(2), \ldots)$ (with $n = 1$) is not causal. For $A = \mathbb{R}$, standard operations from the stream calculus such as pointwise stream addition, shuffle product and shuffle product are all causal.

The above notion of causal functions (with a finite set of arguments V) agrees with the standard notion of causal stream functions (e.g., [12]). Our notion of causal *algebras* generalises it from single functions to algebras for arbitrary functors. This includes polynomial functors modelling a signature. For $A = \mathbb{R}$, the algebra $\alpha \colon \mathcal{P}_\omega(B_\omega) \to B_\omega$ for the finite powerset functor $\mathcal{P}_\omega$, defined by $\alpha(S)(n) = \min\{\sigma(n) \mid \sigma \in S\}$ is a causal algebra which is not a causal function. The algebra $\beta \colon \mathcal{P}_\omega(B_\omega) \to B_\omega$ given by $\beta(S)(n) = \sum_{\sigma \in S} \sigma(n)$ is *not* causal according to Definition 6.1. Intuitively, $\beta(\{\sigma, \tau\})(i)$ depends on equality of σ and τ, since addition of real numbers is not idempotent.

Example 6.2. For the functor $BX = 2 \times X^A$, $B_\omega = \mathcal{P}(A^*)$ is the set of languages over A (Example 2.2). Given languages L and K, we write $L \equiv_i K$ if L and K

contain the same words of length below i. A function $\alpha\colon (\mathcal{P}(A^*))^n \to \mathcal{P}(A^*)$ is causal iff for all languages $L_1, \ldots, L_n, K_1, \ldots, K_n$: if $L_j \equiv_i K_j$ for all $j \leq n$ then $\alpha(L_1, \ldots, L_n) \equiv_i \alpha(K_1, \ldots, K_n)$. For instance, union, concatenation, Kleene star, and shuffle of languages are all causal. An example of a causal algebra that is not a causal function is $\alpha\colon \mathcal{P}(\mathcal{P}(A^*)) \to \mathcal{P}(A^*)$ defined by union.

The following result connects causal algebras to natural transformations of the form $F\bar{B} \Rightarrow \bar{B}$ (which, from Section 4, form a category $\mathcal{K}(\bar{B})$).

Theorem 6.1. *Let $B, F\colon \mathsf{Set} \to \mathsf{Set}$ be functors, and suppose B is ω-continuous. The category $\mathsf{causal}(B)$ of causal algebras is isomorphic to the category $\mathcal{K}(\bar{B})$. Concretely, there is a one-to-one correspondence*

$$\frac{\alpha\colon F\bar{B} \Rightarrow \bar{B}}{\alpha_\omega\colon FB_\omega \to B_\omega \ \textit{causal}}$$

From top to bottom, this is given by evaluation at ω. Moreover, we have $\beta \circ \kappa_{\bar{B}} = \alpha$ iff $\beta_\omega \circ \kappa_{B_\omega} = \alpha_\omega$ for any $\alpha\colon F\bar{B} \Rightarrow \bar{B}$, $\beta\colon G\bar{B} \Rightarrow \bar{B}$ and $\kappa\colon F \Rightarrow G$.

By the above theorem, the universal property of the codensity monad amounts to the following property of causal algebras.

Corollary 6.1. *Suppose $B\colon \mathsf{Set} \to \mathsf{Set}$ is ω-continuous. Let ϵ be the counit of $\mathsf{C}_{\bar{B}}$. Then ϵ_ω is final in $\mathsf{causal}(B)$, i.e., for every causal algebra $\alpha\colon FB_\omega \to B_\omega$, there is a unique natural transformation $\hat{\alpha}\colon F \Rightarrow \mathsf{C}_{\bar{B}}$ such that $\epsilon_\omega \circ \hat{\alpha}_{B_\omega} = \alpha$.*

$$\begin{tikzcd}
FB_\omega \arrow[rr, "\hat{\alpha}_{B_\omega}"] \arrow[dr, "\alpha"'] & & \mathsf{C}_{\bar{B}}B_\omega \arrow[dl, "\epsilon_\omega"] \\
& B_\omega &
\end{tikzcd}$$

By Lemma 4.2 and Lemma 6.1, we obtain the following concrete description of the codensity monad $\mathsf{C}_{\bar{B}}$ of the final sequence of a Set endofunctor B, as a *functor of causal functions.*

Theorem 6.2. *Let $B\colon \mathsf{Set} \to \mathsf{Set}$ be an ω-continuous functor. The codensity monad $\mathsf{C}_{\bar{B}}$ of the final sequence of B is given by*

$$\mathsf{C}_{\bar{B}}(X) = \{\alpha\colon B_\omega^X \to B_\omega \mid \alpha \textit{ is a causal function}\}\,,$$
$$\mathsf{C}_{\bar{B}}(h\colon X \to Y)(\alpha) = \lambda f.\, \alpha(f \circ h)\,,$$

and, for the counit $\epsilon\colon \mathsf{C}_{\bar{B}}\bar{B} \Rightarrow \bar{B}$, we have $\epsilon_\omega(\alpha\colon B_\omega^{B\omega} \to B_\omega) = \alpha(\mathsf{id}_{B_\omega})$.

Hence, the codensity monad of the final sequence of the functor $X \mapsto A \times X$ of stream systems maps a set X to the set of all causal stream functions with $|X|$ arguments. Similarly for the functor $X \mapsto 2 \times X^A$: we obtain a functor of causal functions on languages.

7 Companion of a Set functor

The previous sections gives us a concrete understanding of the codensity monad of the final sequence of a Set functor in terms of causal functions, and Theorem 5.1 provides us with a sufficient condition for this codensity monad to be the companion. We now focus on several applications of these results.

A rather general class of functors that satisfy the hypotheses of Theorem 5.1 is given by the *polynomial functors*. Automata, stream systems, Mealy and Moore machines, various kinds of trees, and many more are all examples of coalgebras for polynomial functors (e.g., [15]). A functor $B\colon \mathsf{Set} \to \mathsf{Set}$ is called polynomial (in a single variable) if it is isomorphic to a functor of the form

$$X \mapsto \coprod_{a \in A} X^{B_a}$$

for some A-indexed collection $(B_a)_{a \in A}$ of sets. As explained in [11, 1.18], a Set functor B is polynomial if and only if it preserves connected limits. This implies existence and preservation by B of the codensity monad of $\bar{B}$, as required by Theorem 5.1 (see the appendix for details).

Lemma 7.1. *If $B\colon \mathsf{Set} \to \mathsf{Set}$ is polynomial, then it satisfies the hypotheses of Theorem 5.1.*

As a consequence, if B is polynomial, the functor of causal functions in Theorem 6.2 is the companion of B.

7.1 Solving equations via causal algebras

As explained in the introduction, a distributive law of F over B allows one to solve systems of equations, formalised in terms of BF-coalgebras, leading to an expressive coinductive definition technique. This approach is formally supported by a solution theorem, stated for the companion in Corollary 3.1. Based on the characterisation of the companion in terms of causal algebras, we obtain a new, simplified solution theorem: it does not mention distributive laws at all, but is stated purely in terms of causal algebras.

Theorem 7.1. *Let $B\colon \mathsf{Set} \to \mathsf{Set}$ be a polynomial functor, with final coalgebra (B_ω, ζ). Let $\alpha\colon FB_\omega \to B_\omega$ be a causal algebra. For every $f\colon X \to BFX$ there is a unique $f^\dagger\colon X \to Z$ such that the following diagram commutes.*

$$\begin{tikzcd}
X \arrow[rr, "f^\dagger"] \arrow[d, "f"'] & & B_\omega \arrow[d, "\zeta"] \\
BFX \arrow[r, "BFf^\dagger"'] & BFB_\omega \arrow[r, "B\alpha"'] & BB_\omega
\end{tikzcd}$$

Example 7.1. For the functor $BX = A \times X$, B_ω is the set of streams. Take $SX = X^2$ for F, and consider the coalgebra $f\colon 1 \to BS1$ with $1 = \{*\}$, defined by $* \mapsto (1, (*, *))$. Pointwise addition is a causal function on streams, modelled by an algebra on B_ω for the functor S. By Theorem 7.1 we obtain a unique solution $\sigma \in B_\omega$, satisfying $\sigma_0 = 1$ and $\sigma' = \sigma \oplus \sigma$. Similarly, the shuffle product of streams is causal, so that by applying Theorem 7.1 with that algebra and the same coalgebra f we obtain a unique stream σ satisfying $\sigma_0 = 1$, $\sigma' = \sigma \otimes \sigma$.

As explained in the Introduction, this method also allows one to define functions on streams. For instance, for the shuffle product, define a BS-coalgebra $f\colon (B_\omega)^2 \to BS(B_\omega)^2$, by $f(\sigma, \tau) = (\sigma_0 \times \tau_0, ((\sigma', \tau), (\tau, \sigma'))$. Since addition of streams is causal, by Theorem 7.1 there is a unique $f^\dagger\colon B_\omega \times B_\omega \to B_\omega$ such that $f^\dagger(\sigma,\tau)(0) = \sigma(0) \times \tau(0)$ and $(f^\dagger(\sigma,\tau))' = (f^\dagger(\sigma',\tau) \oplus f^\dagger(\sigma,\tau'))$, matching the definition given in the Introduction (2). Notice that not every function defined in this way is causal; for instance, it is easy to define even (see Example 6.1), even with the standard coinduction principle (i.e., where $F = \mathsf{Id}$ and $\alpha = \mathsf{id}$).

Example 7.2. Consider the functor $BX = 2 \times X^A$, whose final coalgebra consists of the set $\mathcal{P}(A^*)$ of languages. A $B\mathcal{P}$-coalgebra $f\colon X \to 2 \times (\mathcal{P}(X))^A$ is a non-deterministic automaton. Taking the causal algebra $\alpha\colon \mathcal{P}(\mathcal{P}(A^*)) \to \mathcal{P}(A^*)$ defined by union, the unique map $f^\dagger\colon X \to \mathcal{P}(A^*)$ from Theorem 7.1 is the usual language semantics of non-deterministic automata.

In [44], a context-free grammar (in Greibach normal form) is modelled as a $B\mathcal{P}^\star$-coalgebra $f\colon X \to 2 \times (\mathcal{P}(X)^*)^A$, and its semantics is defined operationally by turning f into a deterministic automaton over $\mathcal{P}(X^*)$. In [35] this operational view is related to the semantics of CFGs in terms of language equations. Consider the causal algebra $\alpha\colon \mathcal{P}(\mathcal{P}(A^*)^*) \to \mathcal{P}(A^*)$ defined by union and language composition: $\alpha(S) = \bigcup_{L_1 \dots L_k \in S} L_1 L_2 \dots L_k$. By Theorem 7.1, any context-free grammar f has a unique solution in languages, which is the semantics of CFGs in the usual sense. As such, we obtain an elementary coalgebraic semantics of CFGs that does not require us to relate it to an operational semantics.

7.2 Causal algebras and distributive laws

Another application of the fact that the codensity monad is the companion is that the final causal algebra in Corollary 6.1 is, by Proposition 5.1, the algebra induced by a distributive law. Hence, *any* causal algebra is "definable" by a distributive law, in the sense that it factors as a (component of a) natural transformation followed by the algebra induced by a distributive law.

More precisely, suppose $B\colon \mathsf{Set} \to \mathsf{Set}$ has a final coalgebra (Z, ζ). We say an algebra $\alpha\colon FZ \to Z$ is *definable by a distributive law over B* if there exists a distributive law $\lambda\colon GB \Rightarrow BG$ with induced algebra $\beta\colon GZ \to Z$ and a natural transformation $\kappa\colon F \Rightarrow G$ such that the following commutes:

$$\begin{tikzcd} FZ \arrow[rr, "\kappa_Z"] \arrow[dr, "\alpha"'] & & GZ \arrow[dl, "\beta"] \\ & Z & \end{tikzcd}$$

Theorem 7.2. *Let $B\colon \mathsf{Set} \to \mathsf{Set}$ be polynomial. An algebra $\alpha\colon FB_\omega \to B_\omega$ is causal if and only if it is definable by a distributive law over B.*

Since the functors for stream systems and automata are polynomial, as a special case of Theorem 7.2 we obtain that a stream function, or a function on languages, is causal if and only if it is definable by a distributive law.

In [12], a similar result is shown concretely for causal stream functions, and this is extended to languages in [34]. In both cases, very specific presentations of distributive laws for the systems at hand are used to present the distributive law based on a "syntax", which however is not too clearly distinguished from the semantics: it consists of a single operation symbol for every causal function. In our case, in the proof of Theorem 7.2, we use the *companion*, which consists of the actual functions rather than a syntactic representation. Indeed, the setting of Theorem 7.2 applies more abstractly to *all* causal algebras, not just causal functions. However, it remains an intriguing question how to obtain a concrete syntactic characterisation of a distributive law for a given causal algebra.

7.3 Soundness of up-to techniques

The *contextual closure* of an algebra is one of the most powerful up-to techniques, which allows one to exploit algebraic structure in bisimulation proofs. In [7], it is shown that the contextual closure is sound (compatible) on any bialgebra for a distributive law. Here, we move away from distributive laws and give an elementary condition for soundness of the contextual closure on the final coalgebra: that the algebra under consideration is causal. In fact, we prove that this implies that the contextual closure lies below the companion, which not only gives soundness, but also allows to combine it with other up-to techniques.

Due to space limitations, we can not fully explain the relevant definitions, and refer to [7] for details. Bisimulations on a B-coalgebra (X, f) are the post-fixed points of a monotone function $b_f\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ on the lattice Rel_X of relations on X, defined by $b_f(R) = f^* \circ \mathsf{Rel}(B)(R)$. Here $\mathsf{Rel}(B)$ is the *relation lifting* of B, and f^* is inverse image along $f \times f$, see, e.g., [15]. Contextual closure $\mathsf{ctx}_\alpha\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ with respect to an algebra $\alpha\colon FX \to X$ is defined dually by $\mathsf{ctx}_\alpha(R) = \coprod_\alpha \circ \mathsf{Rel}(F)(R)$, where $\coprod_\alpha$ is direct image along $\alpha \times \alpha$.

Theorem 7.3. *Let $B\colon \mathsf{Set} \to \mathsf{Set}$ be polynomial, and (B_ω, ζ) a final B-coalgebra. Let t_ζ be the companion of b_ζ. For any causal algebra $\alpha\colon FB_\omega \to B_\omega$: $\mathsf{ctx}_\alpha \leq t_\zeta$.*

This implies that one can safely use the contextual closure for *any* causal algebra, such as union, concatenation and Kleene star of languages, or product and sum of streams. Endrullis et al. [9] prove the soundness of *causal contexts* in combination with other up-to techniques, for equality of streams. The soundness of causal algebras for streams is a special case of Theorem 7.3, but the latter provides more: being below the companion, it is possible to compose it to other such functions to obtain combined up-to techniques in a modular fashion, cf. [31].

8 Abstract GSOS

To obtain expressive specification formats, Turi and Plotkin [42] use natural transformations of the form $\lambda\colon F(B \times \mathsf{Id}) \Rightarrow BF^*$, where F^* is the free monad for F. These are the so-called *abstract GSOS specifications*. We conclude this article by showing that they are actually equally expressive as plain distributive laws of a functor F over B.

If B has a final coalgebra (Z, ζ), then any abstract GSOS specification $\lambda\colon F(B \times \mathsf{Id}) \Rightarrow BF^*$ defines an algebra $\alpha\colon FZ \to Z$ on it, which is the unique algebra making the following diagram commute.

$$\begin{tikzcd}
FZ \arrow[r, "F\langle \zeta, \mathsf{id}\rangle"] \arrow[d, "\alpha"'] & F(B \times \mathsf{Id})Z \arrow[r, "\lambda_Z"] & BF^*Z \arrow[d, "B\alpha^*"] \\
Z \arrow[rr, "\zeta"'] & & BZ
\end{tikzcd}$$

Here α^* is the Eilenberg-Moore algebra for the free monad corresponding to α. Intuitively, this algebra gives the interpretation of the operations defined by λ.

Like plain distributive laws (Lemma 5.1), abstract GSOS specifications induce natural transformations of the form $F\bar{B} \Rightarrow \bar{B}$.

Lemma 8.1. *For every $\lambda\colon F(B \times \mathsf{Id}) \Rightarrow BF^*$ there is a unique $\alpha\colon F\bar{B} \Rightarrow \bar{B}$ such that for all $i \in \mathsf{Ord}$: $\alpha_{i+1} = B\alpha_i^* \circ \lambda_{B_i} \circ F\langle \mathsf{id}, B_{i+1,i}\rangle$. Moreover, if $B_{k+1,k}$ is an isomorphism for some k, then α_k is the algebra induced by λ on the final coalgebra.*

This places abstract GSOS specifications within the framework of the companion, constructed via the codensity monad of the final sequence $\bar{B}$. Whenever that construction applies (e.g., for polynomial functors), any algebra defined by an abstract GSOS is thus already definable by a plain distributive law over B.

Theorem 8.1. *Suppose $B\colon \mathcal{C} \to \mathcal{C}$ satisfies the conditions of Theorem 5.1. Every algebra induced on the final coalgebra by an abstract GSOS specification $\lambda\colon F(B \times \mathsf{Id}) \Rightarrow BF^*$ is definable by a distributive law over B (cf. Section 7.2).*

In this sense, abstract GSOS is no more expressive than plain distributive laws. Note, however, that this does involve moving to a different (larger) syntax.

Remark 8.1. Every abstract GSOS specification $\lambda\colon F(B \times \mathsf{Id}) \Rightarrow BF^*$ corresponds to a unique distributive law $\lambda^\dagger\colon F^*(B \times \mathsf{Id}) \Rightarrow (B \times \mathsf{Id})F^*$ of the free monad F^* over the (cofree) copointed functor $B \times \mathsf{Id}$, see [23]. The algebra induced by λ decomposes as the algebra induced by $\lambda^\dagger$ and the canonical natural transformation $F \Rightarrow F^*$. This implies that every algebra induced by an abstract GSOS is definable by a distributive law over the copointed functor $B \times \mathsf{Id}$. Theorem 8.1 strengthens this to definability by a distributive law over B.

References

1. M. Abadi and A. D. Gordon. A bisimulation method for cryptographic protocols. *Nord. J. Comput.*, 5(4):267–, 1998.
2. S. Abramsky. The Lazy Lambda Calculus. In *Research Topics in Functional Programming*, pages 65–116. Addison Wesley, 1990.
3. J. Adámek. Free algebras and automata realizations in the language of categories. *Commentationes Mathematicae Universitatis Carolinae*, 15(4):589–602, 1974.
4. M. Barr. Algebraically compact functors. *Journal of Pure and Applied Algebra*, 82(3):211 – 231, 1992.
5. F. Bartels. *On generalised coinduction and probabilistic specification formats*. PhD thesis, CWI, Amsterdam, April 2004.
6. F. Bonchi, D. Petrisan, D. Pous, and J. Rot. Coinduction up-to in a fibrational setting. In *Proc. CSL-LICS*, pages 20:1–20:9. ACM, 2014.
7. F. Bonchi, D. Petrişan, D. Pous, and J. Rot. A general account of coinduction up-to. *Acta Informatica*, pages 1–64, 2016.
8. J.-M. Cordier and T. Porter. *Shape theory: categorical methods of approximation.* Mathematics and its Applications. Ellis Horwood, 1989. Reprinted by Dover, 2008.
9. J. Endrullis, D. Hendriks, and M. Bodin. Circular coinduction in coq using bisimulation-up-to techniques. In *Proc. Proc. ITP*, volume 7998 of *LNCS*, pages 354–369. Springer, 2013.
10. C. Fournet, J. Lévy, and A. Schmitt. An asynchronous, distributed implementation of mobile ambients. In *Proc. IFIP TCS*, volume 1872 of *LNCS*, pages 348–364. Springer, 2000.
11. N. Gambino and J. Kock. Polynomial functors and polynomial monads. In *Proc. Mathematical proceedings of the cambridge philosophical society*, volume 154, pages 153–192. Cambridge Univ Press, 2013.
12. H. H. Hansen, C. Kupke, and J. Rutten. Stream differential equations: Specification formats and solution methods. *CoRR*, abs/1609.08367, 2016.
13. C. Hur, G. Neis, D. Dreyer, and V. Vafeiadis. The power of parameterization in coinductive proof. In *Proc. POPL*, pages 193–206. ACM, 2013.
14. B. Jacobs. Distributive laws for the coinductive solution of recursive equations. *Inf. Comput.*, 204(4):561–587, 2006.
15. B. Jacobs. *Introduction to coalgebra. Towards mathematics of states and observations*. Draft, 2014.
16. A. Jeffrey and J. Rathke. A theory of bisimulation for a fragment of concurrent ML with local names. *Theoretical Computer Science*, 323(1-3):1–48, 2004.
17. B. Klin. Bialgebras for structural operational semantics: An introduction. *Theoretical Computer Science*, 412(38):5043–5069, 2011.
18. B. Klin and B. Nachyla. Presenting morphisms of distributive laws. In *Proc. Proc. CALCO*, volume 35 of *LIPIcs*, pages 190–204. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015.
19. B. Knaster. Un théorème sur les fonctions d'ensembles. *Annales de la Socit Polonaise de Mathmatiques*, 6:133–134, 1928.
20. S. M. Lane. *Categories for the working mathematician.* Springer, 1998.
21. T. Leinster. Codensity and the ultrafilter monad. *Theory and Applications of Categories*, 28(13):332–370, 2013.
22. M. Lenisa, J. Power, and H. Watanabe. Distributivity for endofunctors, pointed and co-pointed endofunctors, monads and comonads. *Electr. Notes Theor. Comput. Sci.*, 33:230–260, 2000.

23. M. Lenisa, J. Power, and H. Watanabe. Category theory for operational semantics. *Theor. Comput. Sci.*, 327(1-2):135–154, 2004.
24. X. Leroy. Formal verification of a realistic compiler. *Communications of the ACM*, 52(7):107–115, 2009.
25. S. Milius, L. S. Moss, and D. Schwencke. Abstract GSOS rules and a modular treatment of recursive definitions. *Logical Methods in Computer Science*, 9(3), 2013.
26. R. Milner. *Communication and Concurrency.* Prentice Hall, 1989.
27. R. Milner, J. Parrow, and D. Walker. A calculus of mobile processes, I/II. *Information and Computation*, 100(1):1–77, 1992.
28. nLab article. Kan extension. `http://ncatlab.org/nlab/show/Kan+extension`, 2016. [Revision 103].
29. J. Parrow and T. Weber. The largest respectful function. *Logical Methods in Computer Science*, 12(2), 2016.
30. D. Pous. Complete lattices and up-to techniques. In *Proc. APLAS*, volume 4807 of *LNCS*, pages 351–366. Springer, 2007.
31. D. Pous. Coinduction all the way up. In *Proc. LICS*, pages 307–316. ACM, 2016.
32. D. Pous and D. Sangiorgi. *Advanced Topics in Bisimulation and Coinduction*, chapter about "Enhancements of the coinductive proof method". Cambridge University Press, 2011.
33. J. Power and H. Watanabe. Combining a monad and a comonad. *Theor. Comput. Sci.*, 280(1-2):137–162, 2002.
34. J. Rot, M. M. Bonsangue, and J. Rutten. Proving language inclusion and equivalence by coinduction. *Inf. Comput.*, 246:62–76, 2016.
35. J. Rot and J. Winter. On language equations and grammar coalgebras for context-free languages. In *Proc. CALCO Early Ideas*, 2013.
36. J. J. M. M. Rutten. A coinductive calculus of streams. *Mathematical Structures in Computer Science*, 15(1):93–147, 2005.
37. D. Sangiorgi. On the bisimulation proof method. *Mathematical Structures in Computer Science*, 8:447–479, 1998.
38. D. Sangiorgi and D. Walker. *The π-calculus: a Theory of Mobile Processes*. Cambridge University Press, 2001.
39. J. Sevcík, V. Vafeiadis, F. Z. Nardelli, S. Jagannathan, and P. Sewell. Compcerttso: A verified compiler for relaxed-memory concurrency. *Journal of the ACM*, 60(3):22, 2013.
40. A. Silva, F. Bonchi, M. Bonsangue, and J. Rutten. Generalizing the powerset construction, coalgebraically. In *Proc. Proc. FSTTCS*, pages 272–283, 2010.
41. A. Tarski. A Lattice-Theoretical Fixpoint Theorem and its Applications. *Pacific Journal of Mathematics*, 5(2):285–309, June 1955.
42. D. Turi and G. D. Plotkin. Towards a mathematical operational semantics. In *Proc. LICS*, pages 280–291. IEEE, 1997.
43. H. Watanabe. Well-behaved translations between structural operational semantics. *Electr. Notes Theor. Comput. Sci.*, 65(1):337–357, 2002.
44. J. Winter, M. M. Bonsangue, and J. J. M. M. Rutten. Coalgebraic characterizations of context-free languages. *Logical Methods in Computer Science*, 9(3), 2013.

A Proof of Theorem 3.1

Proof. It only remains to prove that (T, η, μ) is a monad. For the unit axioms, consider the following diagrams.

$$
\begin{tikzcd}
TB \arrow[r, "T\eta B"] \arrow[dd, "\tau"'] \arrow[dr, "TB\eta"'] & TTB \arrow[r, "\mu B"] \arrow[d, "T\tau"] & TB \arrow[dd, "\tau"] \\
 & TBT \arrow[d, "\tau T"] & \\
BT \arrow[r, "BT\eta"'] & BTT \arrow[r, "B\mu"'] & BT
\end{tikzcd}
\qquad
\begin{tikzcd}
TB \arrow[r, "\eta TB"] \arrow[dd, "\tau"'] & TTB \arrow[r, "\mu B"] \arrow[d, "T\tau"] & TB \arrow[dd, "\tau"] \\
 & TBT \arrow[d, "\tau T"] & \\
BT \arrow[r, "B\eta T"'] \arrow[ur, "\eta BT"] & BTT \arrow[r, "B\mu"'] & BT
\end{tikzcd}
$$

Both diagrams commute, by definition of η (the triangles), definition of μ (the rectangles) and by naturality (the rest). We obtain $\mu \circ T\eta = \mathsf{id} = \mu \circ \eta_T$ by uniqueness of morphisms to (T, τ).

For the multiplication, we have the following diagrams:

$$
\begin{tikzcd}
TTTB \arrow[r, "T\mu B"] \arrow[d, "TT\tau"'] & TTB \arrow[r, "\mu B"] \arrow[dd, "T\tau"] & TB \arrow[ddd, "\tau"] \\
TTBT \arrow[d, "T\tau T"'] & & \\
TBTT \arrow[r, "TB\mu"] \arrow[d, "\tau TT"'] & TBT \arrow[d, "\tau T"] & \\
BTTT \arrow[r, "BT\mu"'] & BTT \arrow[r, "B\mu"'] & BT
\end{tikzcd}
\qquad
\begin{tikzcd}
TTTB \arrow[r, "\mu TB"] \arrow[d, "TT\tau"'] & TTB \arrow[r, "\mu B"] \arrow[d, "T\tau"] & TB \arrow[ddd, "\tau"] \\
TTBT \arrow[r, "\mu BT"] \arrow[d, "T\tau T"'] & TBT \arrow[dd, "\tau T"] & \\
TBTT \arrow[d, "\tau TT"'] & & \\
BTTT \arrow[r, "B\mu T"'] & BTT \arrow[r, "B\mu"'] & BT
\end{tikzcd}
$$

The rectangles commute by definition of μ, the squares by naturality. It follows by uniqueness of morphisms to (T, τ) that $\mu \circ \mu_T = \mu \circ T\mu$. $\square$

B Proof of Theorem 3.2

Proof. Let (X, f) be a B-coalgebra. Write $\hat{X}$ for the constant-to-X functor, and $\hat{f}$ for the constant-to-f distributive law of $\hat{X}$ over B. By finality of the companion, there exists a unique natural transformation $\lambda : \hat{X} \Rightarrow T$ such that $B\lambda \circ \hat{f} = \tau \circ \lambda B$. One checks easily that λ_0 is the unique coalgebra homomorphism from (X, f) to $(T0, \tau_0 \circ T!_{B0})$.

To prove that μ_0 is the algebra induced by the companion, it suffices to prove that $(\tau_0 \circ T!_{B0}) \circ \mu_0 = B\mu_0 \circ \tau_{T0} \circ T(\tau_0 \circ T!_{B0})$, which follows from naturality of μ and the fact that τ is a distributive law of a monad over B. $\square$

C Proof of Proposition 3.1

Proof. By Theorem 3.2, $\tau_0 \circ T!_{B0} \colon T0 \to BT0$ is a final B-coalgebra. By definition of the algebra induced on the final coalgebra by λ, and uniqueness of morphisms into final coalgebras, it suffices to prove that the following diagram commutes.

$$\begin{tikzcd}
FT0 \arrow[r, "\bar{\lambda}_{T0}"] \arrow[d, "FT!_{B0}"'] & TT0 \arrow[r, "\mu_0"] \arrow[d, "TT!_{B0}"] & T0 \arrow[d, "T!_{B0}"] \\
FTB0 \arrow[r, "\bar{\lambda}_{TB0}"] \arrow[d, "F\tau_0"'] & TTB0 \arrow[r, "\mu_{B0}"] \arrow[d, "T\tau_0"] & TB0 \arrow[dd, "\tau_0"] \\
FBT0 \arrow[r, "\bar{\lambda}_{BT0}"] \arrow[d, "\lambda_{T0}"'] & TBT0 \arrow[d, "\tau_{T0}"] & \\
BFT0 \arrow[r, "B\bar{\lambda}_{T0}"'] & BTT0 \arrow[r, "B\mu_0"'] & BT0
\end{tikzcd}$$

Everything commutes: clockwise starting from the top right by naturality, definition of μ, the fact that $\bar{\lambda}$ is a morphism from (F, λ) to (T, τ), and twice naturality. $\square$

D Proof of Lemma 5.1

Proof. This natural transformation is completely determined by the successor case given in the definition; on a limit ordinal j, B_j is a limit, and naturality requires it to be defined as the unique arrow $\alpha_j \colon FB_j \to B_j$ such that

$$\begin{tikzcd}
FB_j \arrow[r, dashed, "\alpha_j"] \arrow[d, "FB_{j,i}"'] & B_j \arrow[d, "B_{j,i}"] \\
FB_i \arrow[r, "\alpha_i"'] & B_i
\end{tikzcd}$$

commutes, for all $i < j$.

For naturality, we have to prove that the relevant square (as above) commutes for all $i \leq j$. For $i = j$, this follows since $B_{j,j} = \mathsf{id}_{B_j}$ by definition. We prove that the square commutes for any $i < j$, by induction on j. The case that j is a limit ordinal follows immediately from the definition of α_j. Now suppose that, for any i with $i < j$, the square commutes for i, j. Then it also commutes for $i + 1 < j + 1$:

$$\begin{tikzcd}
FB_{j+1} = FBB_j \arrow[r, "\lambda_{B_j}"] \arrow[d, "FB_{j+1,i+1}=FBB_{j,i}"'] & BFB_j \arrow[r, "B\alpha_j"] \arrow[d, "BFB_{j,i}"] & BB_j = B_{j+1} \arrow[d, "B_{j+1,i+1}=BB_{j,i}"] \\
FB_{i+1} = FBB_i \arrow[r, "\lambda_{B_i}"] & BFB_i \arrow[r, "B\alpha_i"] & BB_i = B_{i+1}
\end{tikzcd}$$

by naturality and assumption. For i a limit ordinal, consider the following diagram:

$$\begin{tikzcd} FB_j \arrow[r, "FB_{j,i}"] \arrow[d, "\alpha_j"'] & FB_i \arrow[r, "FB_{i,l}"] \arrow[d, "\alpha_i"] & FB_l \arrow[d, "\alpha_l"] \\ B_j \arrow[r, "B_{j,i}"'] & B_i \arrow[r, "B_{i,l}"'] & B_l \end{tikzcd}$$

For all $l < i$, the outer rectangle commutes by the inductive hypothesis, and the right square by definition of α_i on the limit ordinal i. Since B_i is a limit with projections $B_{i,l}$ for $l \leq i$, it follows that the square on the left commutes, as desired.

For the second point in the statement: if $B_{k+1,k}\colon B_{k+1} \to B_k$ is an isomorphism, then $B^{-1}_{k+1,k}\colon B_k \to B(B_{k+1})$ is a final B-coalgebra. Consider the following diagram:

$$\begin{tikzcd} FB_k \arrow[r, "\alpha_k"] \arrow[d, "FB^{-1}_{k+1,k}"'] & B_k \arrow[dd, "B^{-1}_{k+1,k}"] \\ FBB_k \arrow[d, "\lambda_{B_k}"'] \arrow[dr, "\alpha_{k+1}"] & \\ BFB_k \arrow[r, "B\alpha_k"'] & BB_k \end{tikzcd}$$

The triangle commutes by definition of α, and the shape above it by naturality and the fact that $B_{k+1,k}$ is an isomorphism. It follows that α_k is the algebra induced on the final coalgebra by λ. $\square$

E Proof of Theorem 5.1

Proof. By assumption, $(B\mathsf{C}_{\bar{B}}, B\epsilon)$ is the right Kan extension of $B\bar{B}$ along $\bar{B}$. This means that for all $\alpha\colon H\bar{B} \Rightarrow B\bar{B}$, there exists a unique $\hat{\alpha}\colon H \Rightarrow B\mathsf{C}_{\bar{B}}$ such that $\alpha = B\epsilon \circ \hat{\alpha}_{\bar{B}}$. We use this universal property to define the natural transformation τ, choosing $H = \mathsf{C}_{\bar{B}}B$.

To this end, consider the functor $S\colon \mathsf{Ord}^{\mathsf{op}} \to \mathsf{Ord}^{\mathsf{op}}$ defined by $S(i) = i + 1$. The following diagram commutes:

$$\begin{tikzcd} \mathsf{Ord}^{\mathsf{op}} \arrow[r, "\bar{B}"] \arrow[d, "S"'] & \mathcal{C} \arrow[d, "B"] \\ \mathsf{Ord}^{\mathsf{op}} \arrow[r, "\bar{B}"'] & \mathcal{C} \end{tikzcd} \tag{6}$$

It simply expresses that $B_{j+1,i+1} = BB_{j,i}$ for all $i \leq j$, which holds by definition of the final sequence. As a consequence, there is the natural transformation on the top row of the diagram below:

$$
\begin{tikzcd}
\mathsf{C}_{\bar{B}} B \bar{B} \arrow[r, equal] \arrow[d, Rightarrow, "\tau \bar{B}"'] & \mathsf{C}_{\bar{B}} \bar{B} S \arrow[r, Rightarrow, "\epsilon S"] & \bar{B} S \arrow[r, equal] & B \bar{B} \\
B \mathsf{C}_{\bar{B}} \bar{B} \arrow[urrr, Rightarrow, "B\epsilon"'] & & & .
\end{tikzcd}
\tag{7}
$$

By the universal property of $(B\mathsf{C}_{\bar{B}}, B\epsilon)$ we obtain $\tau \colon \mathsf{C}_{\bar{B}} B \Rightarrow B\mathsf{C}_{\bar{B}}$ as the unique natural transformation making the above diagram (7) commute.

We prove that τ is a distributive law of the codensity monad $(\mathsf{C}_{\bar{B}}, \eta, \mu)$ over B. To this end, recall that η and μ are defined as the unique natural transformations making the following diagrams commute:

$$
\begin{tikzcd}
\bar{B} \arrow[r, Rightarrow, "\eta \bar{B}"] \arrow[dr, equal] & \mathsf{C}_{\bar{B}} \bar{B} \arrow[d, Rightarrow, "\epsilon"] \\
& \bar{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{C}_{\bar{B}} \mathsf{C}_{\bar{B}} \bar{B} \arrow[r, Rightarrow, "\mathsf{C}_{\bar{B}} \epsilon"] \arrow[d, Rightarrow, "\mu \bar{B}"'] & \mathsf{C}_{\bar{B}} \bar{B} \arrow[d, Rightarrow, "\epsilon"] \\
\mathsf{C}_{\bar{B}} \bar{B} \arrow[r, Rightarrow, "\epsilon"'] & \bar{B}
\end{tikzcd}
\tag{8}
$$

For the unit axiom, we need to show commutativity of:

$$
\begin{tikzcd}
B \arrow[r, Rightarrow, "\eta B"] \arrow[dr, Rightarrow, "B\eta"'] & \mathsf{C}_{\bar{B}} B \arrow[d, Rightarrow, "\tau"] \\
& B \mathsf{C}_{\bar{B}}
\end{tikzcd}
$$

which we do by showing that $B\epsilon \circ \tau_{\bar{B}} \circ \eta_{B\bar{B}} = B\epsilon \circ B\eta_{\bar{B}}$; the desired equality then follows from uniqueness (from the universal property of $B\epsilon$). This, in turn, follows from commutativity of:

$$
\begin{tikzcd}
& & \mathsf{C}_{\bar{B}} B \bar{B} \arrow[d, equal] \arrow[ddr, Rightarrow, "\tau \bar{B}"] & \\
B \bar{B} \arrow[urr, Rightarrow, "\eta B \bar{B}"] \arrow[r, equal] \arrow[ddrr, equal] \arrow[dd, Rightarrow, "B\eta \bar{B}"'] & \bar{B} S \arrow[r, Rightarrow, "\eta \bar{B} S"] \arrow[dr, equal] & \mathsf{C}_{\bar{B}} \bar{B} S \arrow[d, Rightarrow, "\epsilon S"] & \\
& & \bar{B} S \arrow[d, equal] & B \mathsf{C}_{\bar{B}} \bar{B} \arrow[dl, Rightarrow, "B\epsilon"] \\
B \mathsf{C}_{\bar{B}} \bar{B} \arrow[rr, Rightarrow, "B\epsilon"'] & & B \bar{B} &
\end{tikzcd}
$$

The two triangles within the big square commute by (8), the upper left triangle and the trapezoid in the square since $\bar{B} S = B\bar{B}$ (see (6)), and the right triangle by definition of τ (see (7)).

For the multiplication, we are to prove commutativity of:

$$
\begin{tikzcd}
\mathsf{C}_{\bar{B}}\mathsf{C}_{\bar{B}}B \arrow[r, Rightarrow, "\mu B"] \arrow[d, Rightarrow, "\mathsf{C}_{\bar{B}}\tau"'] & \mathsf{C}_{\bar{B}}B \arrow[dd, Rightarrow, "\tau"] \\
\mathsf{C}_{\bar{B}}B\mathsf{C}_{\bar{B}} \arrow[d, Rightarrow, "\tau\mathsf{C}_{\bar{B}}"'] & \\
B\mathsf{C}_{\bar{B}}\mathsf{C}_{\bar{B}} \arrow[r, Rightarrow, "B\mu"'] & B\mathsf{C}_{\bar{B}}
\end{tikzcd}
$$

which, in a similar manner as above for the unit, follows from the universal property of $B\epsilon$ and commutativity of the following diagram.

$$
\begin{tikzcd}
\mathsf{C}_{\bar{B}}\mathsf{C}_{\bar{B}}B\bar{B} \arrow[rr, Rightarrow, "\mu B\bar{B}"] \arrow[dr, equal] \arrow[ddd, Rightarrow, "\mathsf{C}_{\bar{B}}\tau\bar{B}"'] & & \mathsf{C}_{\bar{B}}B\bar{B} \arrow[r, Rightarrow, "\tau\bar{B}"] \arrow[d, equal] & B\mathsf{C}_{\bar{B}}\bar{B} \arrow[dddd, Rightarrow, "B\epsilon"] \\
& \mathsf{C}_{\bar{B}}\mathsf{C}_{\bar{B}}\bar{B}S \arrow[r, Rightarrow, "\mu\bar{B}S"] \arrow[d, Rightarrow, "\mathsf{C}_{\bar{B}}\epsilon S"'] & \mathsf{C}_{\bar{B}}\bar{B}S \arrow[d, Rightarrow, "\epsilon S"] & \\
& \mathsf{C}_{\bar{B}}\bar{B}S \arrow[r, Rightarrow, "\epsilon S"'] \arrow[d, equal] & \bar{B}S \arrow[ddr, equal] & \\
\mathsf{C}_{\bar{B}}B\mathsf{C}_{\bar{B}}\bar{B} \arrow[r, Rightarrow, "\mathsf{C}_{\bar{B}}B\epsilon"] \arrow[dd, Rightarrow, "\tau\mathsf{C}_{\bar{B}}\bar{B}"'] & \mathsf{C}_{\bar{B}}B\bar{B} \arrow[d, Rightarrow, "\tau\bar{B}"'] & & \\
& B\mathsf{C}_{\bar{B}}\bar{B} \arrow[rr, Rightarrow, "B\epsilon"] & & B\bar{B} \\
B\mathsf{C}_{\bar{B}}\mathsf{C}_{\bar{B}}\bar{B} \arrow[ur, Rightarrow, "B\mathsf{C}_{\bar{B}}\epsilon"] \arrow[rrr, Rightarrow, "B\mu\bar{B}"'] & & & B\mathsf{C}_{\bar{B}}\bar{B} \arrow[u, Rightarrow, "B\epsilon"']
\end{tikzcd}
$$

The square in the middle commutes by definition of μ (see (8)). The rest commutes, clockwise starting from the north, by the equality $\bar{B}S = B\bar{B}$ (see (6)), twice definition of τ (see (7)), definition of μ (the south), naturality of τ and again definition of τ.

This concludes the proof that τ is a distributive law. We now show that it is the companion of B, i.e., that it is final in the category $\mathsf{DL}(B)$. To this end, let $\lambda\colon FB \Rightarrow BF$ be a distributive law. We need to prove that there exists a unique natural transformation $\hat{\alpha}\colon F \Rightarrow \bar{B}$ which is a morphism from λ to τ, i.e., making the following diagram commute:

$$\begin{tikzcd} FB \arrow[r, Rightarrow, "\hat{\alpha}B"] \arrow[d, Rightarrow, "\lambda"'] & \mathsf{C}_{\bar{B}}B \arrow[d, Rightarrow, "\tau"] \\ BF \arrow[r, Rightarrow, "B\hat{\alpha}"] & B\mathsf{C}_{\bar{B}} \end{tikzcd} \tag{9}$$

For every natural transformation of the form $\alpha \colon F\bar{B} \Rightarrow \bar{B}$, there is a unique $\hat{\alpha} \colon F \Rightarrow \mathsf{C}_{\bar{B}}$ such that $\epsilon \circ \hat{\alpha}_{\bar{B}} = \alpha$, by the universal property of $B\epsilon$. We shall prove that $\hat{\alpha}$ satisfies (9) if and only if α makes the following diagram commute:

$$\begin{tikzcd} FB\bar{B} \arrow[rr, Rightarrow, "\lambda\bar{B}"] \arrow[d, equal] & & BF\bar{B} \arrow[d, Rightarrow, "B\alpha"] \\ F\bar{B}S \arrow[r, Rightarrow, "\alpha S"'] & \bar{B}S \arrow[r, equal] & B\bar{B} \end{tikzcd} \tag{10}$$

By Lemma 5.1, λ induces a unique α making the above diagram commute. Hence, it then follows that $\hat{\alpha}$ is the unique morphism to τ.

By the universal property of $B\epsilon$, (9) commutes if and only if the following equation holds:

$$B\epsilon \circ \tau_{\bar{B}} \circ \hat{\alpha}_{B\bar{B}} = B\epsilon \circ B\hat{\alpha}_{\bar{B}} \circ \lambda_{\bar{B}} \tag{11}$$

Hence, it suffices to prove that (10) is equivalent to (11).

Consider the following diagram:

$$\begin{tikzcd} FB\bar{B} \arrow[rrrr, Rightarrow, "\hat{\alpha}B\bar{B}"] \arrow[dr, equal] \arrow[dddd, Rightarrow, "\lambda\bar{B}"'] & & & & \mathsf{C}_{\bar{B}}B\bar{B} \arrow[dl, equal] \arrow[dddd, Rightarrow, "\tau\bar{B}"] \\ & F\bar{B}S \arrow[rr, Rightarrow, "\hat{\alpha}\bar{B}S"] \arrow[dr, Rightarrow, "\alpha S"'] & & \mathsf{C}_{\bar{B}}\bar{B}S \arrow[dl, Rightarrow, "\epsilon S"] & \\ & & \bar{B}S \arrow[d, equal] & & \\ & & B\bar{B} & & \\ BF\bar{B} \arrow[urr, Rightarrow, "B\alpha"] \arrow[rrrr, Rightarrow, "B\hat{\alpha}\bar{B}"'] & & & & B\mathsf{C}_{\bar{B}}\bar{B} \arrow[ull, Rightarrow, "B\epsilon"'] \end{tikzcd}$$

The two triangles commute by definition of $\hat{\alpha}$, the upper trapezoid by the equality $B\bar{B} = \bar{B}S$, the right trapezoid by definition of τ. The left trapezoid is (10). The equivalence of (10) and (11) follows from a straightforward diagram chase. $\square$

F Proof of Proposition 5.1

Proof. By definition of τ in the proof of Theorem 5.1, we have $B\epsilon_i \circ \tau_{B_i} = \epsilon_{i+1}$ for all i. Hence, by Lemma 5.1, ϵ_k is the algebra induced by τ on the final B-coalgebra.

For the second point, τ is a distributive law of the codensity monad $(\mathsf{C}_{\bar{B}}, \eta, \mu)$ over B. Since $\mathsf{C}_{\bar{B}}$ is the companion of B, η and μ coincide with the natural transformations in Theorem 3.1. By Theorem 3.2, $\mathsf{C}_{\bar{B}}0$ is the carrier of a final coalgebra and μ_0 is the algebra induced on $\mathsf{C}_{\bar{B}}0$. It follows from the first point that μ_0 is isomorphic to ϵ_k. $\square$

G Proof of Theorem 6.1

Proof. Let $\alpha \colon F\bar{B} \Rightarrow \bar{B}$. We need to prove that α_ω is causal; to this end, let $f, g \colon X \to FB_\omega$ be functions such that $B_{\omega,i} \circ f = B_{\omega,i} \circ g$ for some i. Then the following diagram commutes:

$$\begin{tikzcd}
 & FB_\omega \arrow[r, "\alpha_\omega"] \arrow[dr, "FB_{\omega,i}"'] & B_\omega \arrow[dr, "B_{\omega,i}"] & \\
FX \arrow[ur, "Ff"] \arrow[dr, "Fg"'] & & FB_i \arrow[r, "\alpha_i"] & B_i \\
 & FB_\omega \arrow[r, "\alpha_\omega"'] \arrow[ur, "FB_{\omega,i}"] & B_\omega \arrow[ur, "B_{\omega,i}"'] &
\end{tikzcd}$$

by assumption and naturality of α. Hence α_ω is causal.

Next, we show how to define α from a given α_ω. Since B is ω-continuous and any Set endofunctor preserves epimorphims, one can prove by induction that for any $i < \omega$, the map $B_{\omega,i}$ is an epi. We will use that epis in Set split, i.e., every $B_{\omega,i}$ has a right inverse $B_{\omega,i}^{-1}$ with $B_{\omega,i} \circ B_{\omega,i}^{-1} = \mathsf{id}$.

Given $\alpha_\omega \colon FB_\omega \to B_\omega$, define $\alpha \colon F\bar{B} \Rightarrow \bar{B}$ on a component $i < \omega$ by

$$FB_i \xrightarrow{F(B_{\omega,i}^{-1})} FB_\omega \xrightarrow{\alpha_\omega} B_\omega \xrightarrow{B_{\omega,i}} B_i$$

where $B_{\omega,i}^{-1}$ is a right inverse of $B_{\omega,i}$; and, on a component $i \geq \omega$ by

$$FB_i \xrightarrow{FB_{i,\omega}} FB_\omega \xrightarrow{\alpha_\omega} B_\omega \xrightarrow{B_{i,\omega}^{-1}} B_i$$

where $B_{i,\omega}^{-1}$ is the inverse of $B_{i,\omega}$ (which is an isomorphism).

We need to show that α is a natural transformation, and that the correspondence is bijective. For the bijective correspondence, first note that mapping α_ω to α and back trivially yields α_ω again. Conversely, given α, we need to prove that the following diagrams commute for $i < \omega$ (on the left) and $i \geq \omega$ (on the right):

$$\begin{tikzcd} FB_i \arrow[r, "\alpha_i"] \arrow[d, "F(B_{\omega,i}^{-1})"'] & B_i \\ FB_\omega \arrow[r, "\alpha_\omega"'] & B_\omega \arrow[u, "B_{\omega,i}"'] \end{tikzcd} \qquad \begin{tikzcd} FB_i \arrow[r, "\alpha_i"] \arrow[d, "FB_{i,\omega}"'] & B_i \\ FB_\omega \arrow[r, "\alpha_\omega"'] & B_\omega \arrow[u, "B_{i,\omega}^{-1}"'] \end{tikzcd}$$

The case $i < \omega$ follows by naturality of α and since $B_{\omega,i}^{-1}$ is a right inverse of $B_{\omega,i}$, the case $i \geq \omega$ by naturality of α and since $B_{i,\omega}^{-1}$ is a (left) inverse of $B_{i,\omega}$.

It remains to show that α, defined from a given α_ω as above, is natural, using that α_ω is causal. To this end, let $i \leq j$; to prove is that the following diagram commutes:

$$\begin{tikzcd} FB_i \arrow[r, "\alpha_i"] & B_i \\ FB_j \arrow[u, "B_{j,i}"] \arrow[r, "\alpha_j"'] & B_j \arrow[u, "B_{i,j}"'] \end{tikzcd}$$

where α_i, α_j are defined from α_ω as above. We proceed with a case distinction.

If $i, j < \omega$, then the following diagram commutes:

$$\begin{tikzcd} B_j \arrow[r, "B_{j,i}"] \arrow[d, "B_{\omega,j}^{-1}"'] \arrow[dr, equal] & B_i \arrow[r, "B_{\omega,i}^{-1}"] \arrow[dr, equal] & B_\omega \arrow[d, "B_{\omega,i}"] \\ B_\omega \arrow[r, "B_{\omega,j}"] \arrow[rr, bend right, "B_{\omega,i}"'] & B_j \arrow[r, "B_{j,i}"] & B_i \end{tikzcd}$$

since $B_{\omega,i}^{-1}$ and $B_{\omega,j}^{-1}$ are right inverses (for the two triangles) and the final sequence $\bar{B}$ is a functor (for the crescent). By causality of α_ω (and functoriality of $\bar{B}$) we obtain commutativity of:

$$\begin{tikzcd} FB_i \arrow[r, "F(B_{\omega,i}^{-1})"] & FB_\omega \arrow[r, "\alpha_\omega"] & B_\omega \arrow[r, "B_{\omega,i}"] & B_i \\ FB_j \arrow[u, "FB_{j,i}"] \arrow[r, "F(B_{\omega,j}^{-1})"'] & FB_\omega \arrow[r, "\alpha_\omega"'] & B_\omega \arrow[r, "B_{\omega,j}"'] \arrow[ur, "B_{\omega,i}"] & B_j \arrow[u, "B_{j,i}"'] \end{tikzcd}$$

which is what we needed to prove, by definition of α_i,α_j.

If $i < \omega \leq j$, then the following diagram commutes:

$$\begin{tikzcd} B_i \arrow[r, "B_{\omega,i}^{-1}"] & B_\omega \arrow[r, "B_{\omega,i}"] & B_i \\ B_j \arrow[u, "B_{j,i}"] \arrow[rr, "B_{j,\omega}"'] & & B_\omega \arrow[u, "B_{\omega,i}"'] \end{tikzcd}$$

since $B_{\omega,i} \circ B_{\omega,i}^{-1} = \mathsf{id}$, and the final sequence $\bar{B}$ is a functor. Hence, by causality of α we obtain the commutativity of the large inner part in:

$$\begin{tikzcd}
FB_i \arrow[r, "F(B_{\omega,i}^{-1})"] & FB_\omega \arrow[r, "\alpha_\omega"] & B_\omega \arrow[r, "B_{\omega,i}"] & B_i \\
FB_j \arrow[u, "FB_{j,i}"] \arrow[r, "FB_{j,\omega}"'] & FB_\omega \arrow[r, "\alpha_\omega"'] & B_\omega \arrow[ur, "B_{\omega,i}"] \arrow[r, "B_{j,\omega}^{-1}"'] & B_j \arrow[u, "B_{j,i}"']
\end{tikzcd}$$

The triangle commutes by functoriality of $\bar{B}$ and that $B_{j,\omega}^{-1}$ is an inverse of $B_{j,\omega}$.

Finally, if $\omega \leq i \leq j$, then we immediately obtain commutativity of:

$$\begin{tikzcd}
FB_i \arrow[r, "FB_{i,\omega}"] & FB_\omega \arrow[r, "\alpha_\omega"] & B_\omega \arrow[r, "B_{i,\omega}^{-1}"] & B_i \\
FB_j \arrow[u, "FB_{j,i}"] \arrow[ur, "FB_{j,\omega}"'] \arrow[r, "FB_{j,\omega}"'] & FB_\omega \arrow[r, "\alpha_\omega"'] & B_\omega \arrow[ur, "B_{i,\omega}^{-1}"] \arrow[r, "B_{j,\omega}^{-1}"'] & B_j \arrow[u, "B_{j,i}"']
\end{tikzcd}$$

The triangles commute by functoriality of $\bar{B}$ and the fact that $B_{i,\omega}^{-1}$ and $B_{j,\omega}^{-1}$ are inverses of $B_{i,\omega}$ and $B_{j,\omega}$ respectively.

This concludes the one-to-one correspondence between natural transformations α and causal algebras α_ω. We turn to the second correspondence in the statement: the equivalence

$$\begin{tikzcd}
F\bar{B} \arrow[rr, "\kappa\bar{B}"] \arrow[dr, "\alpha"'] & & G\bar{B} \arrow[dl, "\beta"] \\
& \bar{B} &
\end{tikzcd}
\qquad
\begin{tikzcd}
FB_\omega \arrow[rr, "\kappa_{B_\omega}"] \arrow[dr, "\alpha_\omega"'] & & GB_\omega \arrow[dl, "\beta_\omega"] \\
& B_\omega &
\end{tikzcd}$$

for any $\alpha\colon F\bar{B} \Rightarrow \bar{B}$, $\beta\colon G\bar{B} \Rightarrow \bar{B}$ and $\kappa\colon F \Rightarrow G$. From left to right this is trivial; suppose that the right triangle commutes. Let $i < \omega$. By the above, we have $\alpha_i = B_{\omega,i} \circ \alpha_\omega \circ B_{\omega,i}^{-1}$ and $\beta_i = B_{\omega,i} \circ \beta_\omega \circ B_{\omega,i}^{-1}$, for any right inverse ($B_{\omega,i}^{-1}$ of $B_{\omega,i}$. Hence, it suffices to prove that the diagram on the left below commutes:

$$\begin{tikzcd}
FB_i \arrow[rr, "\kappa_{B_i}"] \arrow[d, "F(B_{\omega,i}^{-1})"'] & & GB_i \arrow[d, "G(B_{\omega,i}^{-1})"] \\
FB_\omega \arrow[rr, "\kappa_{B_\omega}"] \arrow[d, "\alpha_\omega"'] & & GB_\omega \arrow[d, "\beta_\omega"] \\
B_\omega \arrow[rr, equal] \arrow[dr, "B_{\omega,i}"'] & & B_\omega \arrow[dl, "B_{\omega,i}"] \\
& B_i &
\end{tikzcd}
\qquad
\begin{tikzcd}
FB_i \arrow[rr, "\kappa_{B_i}"] \arrow[d, "FB_{i,\omega}"'] & & GB_i \arrow[d, "GB_{i,\omega}"] \\
FB_\omega \arrow[rr, "\kappa_{B_\omega}"] \arrow[d, "\alpha_\omega"'] & & GB_\omega \arrow[d, "\beta_\omega"] \\
B_\omega \arrow[rr, equal] \arrow[dr, "B_{i,\omega}^{-1}"'] & & B_\omega \arrow[dl, "B_{i,\omega}^{-1}"] \\
& B_i &
\end{tikzcd}$$

The upper square of the left diagram commutes by naturality of κ, and the lower by assumption. For $i \geq \omega$, we have $\alpha_i = B_{i,\omega}^{-1} \circ \alpha_\omega \circ B_{i,\omega}$ and $\beta_i = B_{i,\omega}^{-1} \circ \beta_\omega \circ B_{i,\omega}$, hence it suffices to prove commutativity of the diagram on the right above. That follows, again, from naturality of κ and the assumption. □

H Proof of Lemma 7.1

We first recall that a category is *connected* if there is a zigzag of morphisms between any two objects X and Y: a finite collection of morphisms of the form

$$X = X_0 \longrightarrow X_1 \longleftarrow X_2 \longrightarrow X_3 \longleftarrow X_4 \longrightarrow \ldots \longleftarrow X_n = Y$$

that is, a path with both directions of arrows possible [20]. A *connected limit* is a limit over a connected category.

Proof. For each $k \geq \omega$, $B_{k,\omega}$ is an isomorphism, which implies that the category $\Delta_X/\bar{B}$ is essentially small for every set X. Hence, the limit

$$\lim\left((\Delta_X/F) \to \mathcal{C} \xrightarrow{G} \mathcal{D}\right) \tag{12}$$

exists for each X, which, by Lemma 4.1, defines the codensity monad $\mathsf{C}_{\bar{B}}$. Since B is polynomial, it preserves connected limits [11, Proposition 1.16]. We show that $\Delta_X/\bar{B}$ is connected: $\Delta_X/\bar{B}$ is inhabited since there is the arrow $!_X : X \to 1$; and for any $f\colon X \to B_i$, there is the arrow $B_{i,1}$ to $!_X : X \to 1$, which is a morphism in $\Delta_X/\bar{B}$ by uniqueness. Hence, B preserves the limits in (12). This implies that B preserves $\mathsf{C}_{\bar{B}}$, which we spell out in detail.

Denote, for a given set X, the limiting cone of (12) by $\{s^X_f : \mathsf{C}_{\bar{B}}X \to B_i\}_{f \in B_i^X, i \in \mathsf{Ord}}$. The counit of the codensity monad is defined by $\epsilon_i = s^{BX}_{\mathsf{id}_{B_i}}$ (see, e.g., [28,20]). Since B preserves these limits, for each X, we have that $\{Bs^X_f : B\mathsf{C}_{\bar{B}}X \to BB_i\}_{f \in B_i^X, i \in \mathsf{Ord}}$ is the limit

$$\lim\left((\Delta_X/F) \to \mathcal{C} \xrightarrow{G} \mathcal{D} \xrightarrow{B} \mathcal{D}\right) .$$

Hence, by Lemma 4.1, $B\mathsf{C}_{\bar{B}}$ is a right Kan extension of $B\bar{B}$ along $\bar{B}$, with counit defined on $i \in \mathsf{Ord}$ by $Bs^{BX}_{\mathsf{id}_{B_i}} = B\epsilon_i$ as desired. □

I Proof of Theorem 7.1

Consider the codensity monad $\mathsf{C}_{\bar{B}}$, with counit ϵ; by Lemma 7.1, it satisfies the hypotheses of Theorem 5.1. By Corollary 6.1, there is a unique natural transformation $\hat{\alpha}\colon F \Rightarrow \mathsf{C}_{\bar{B}}$ such that $\epsilon_\omega \circ \hat{\alpha}_{B_\omega} = \alpha$. By Theorem 5.1, there is a distributive law τ of the monad $(\mathsf{C}_{\bar{B}}, \eta, \mu)$ over $\bar{B}$. By Proposition 5.1, ϵ_ω is the algebra induced by τ on the final coalgebra.

Let $f\colon X \to BFX$. By Corollary 3.1, there exists a unique $f^\dagger$ making the outside of the following diagram commute (for f in the statement of the corollary we take $B\hat{\alpha}_X \circ f$).

$$\begin{tikzcd}
X \arrow[rr, "f^\dagger"] \arrow[d, "f"'] & & B_\omega \arrow[d, "\zeta"] \\
BFX \arrow[r, "BFf^\dagger"] \arrow[d, "B\hat{\alpha}_X"'] & BFB_\omega \arrow[r, "B\alpha"] \arrow[d, "B\hat{\alpha}_{B_\omega}"'] & BB_\omega \arrow[d, equal] \\
B\mathsf{C}_{\bar{B}}X \arrow[r, "B\mathsf{C}_{\bar{B}}f^\dagger"'] & B\mathsf{C}_{\bar{B}}B_\omega \arrow[r, "B\epsilon_\omega"'] & BB_\omega
\end{tikzcd}$$

The lower left square commutes by naturality, and lower right square by definition of $\hat{\alpha}$. Thus the outside of the diagram commutes if and only if the inner rectangle commutes. It follows that $f^\dagger$ is the unique map making the rectangle commute, which is what we needed to prove. □

J Proof of Theorem 7.2

Proof. First of all, notice that B_ω is indeed the carrier of a final coalgebra, since any polynomial functor is ω-continuous. For the implication from right to left, by Lemma 5.1, a distributive law $\lambda\colon GB \Rightarrow BG$ defines $\beta\colon G\bar{B} \Rightarrow \bar{B}$ such that β_ω is the algebra induced by λ. We need to prove that, given a natural transformation $\kappa\colon F \Rightarrow G$, the algebra $\alpha = \kappa_{B_\omega} \circ \beta_\omega$ is causal. But this follows by Theorem 6.1, since $\kappa_B \circ \beta$ is a natural transformation.

For the converse, let $\alpha\colon FB_\omega \to B_\omega$ be causal. By Corollary 6.1, there is a natural transformation $\hat{\alpha}\colon F \Rightarrow \mathsf{C}_{\bar{B}}$ such that $\alpha = \epsilon_\omega \circ \hat{\alpha}_{B_\omega}$. By Lemma 7.1, B satisfies the hypotheses of Theorem 5.1, and hence by Proposition 5.1, ϵ_ω is the algebra induced by a distributive law (the companion). Hence α is definable by a distributive law over B. □

K Proof of Theorem 7.3

We first prove a general property of algebras and the contextual closure.

Lemma K.1. *Let X be a set, and consider algebras α, β and a natural transformation κ as below:*

$$\begin{tikzcd}
FX \arrow[rr, "\kappa_X"] \arrow[dr, "\alpha"'] & & GX \arrow[dl, "\beta"] \\
& X &
\end{tikzcd}$$

Then $\mathsf{ctx}_\alpha \leq \mathsf{ctx}_\beta$.

Proof. This proof relies on the setting and terminology of [7], which we do not fully recall here. The natural transformation κ lifts to a natural transformation $\mathsf{Rel}(\kappa)\colon \mathsf{Rel}(F) \Rightarrow \mathsf{Rel}(G)$, see [15, Exercise 4.4.6]. It follows from a general property of fibrations (see [7, Lemma 14.5]) that there exists a natural transformation of the form $\coprod_{\kappa_X} \circ \mathsf{Rel}(F) \Rightarrow \mathsf{Rel}(G)\colon \mathsf{Rel}_X \to \mathsf{Rel}_{GX}$. Hence, we obtain a natural transformation

$$\begin{aligned}
\mathsf{ctx}_\alpha &= \textstyle\coprod_{\alpha_X} \circ \mathsf{Rel}(F) \\
&= \textstyle\coprod_{\beta\circ\kappa_X} \circ \mathsf{Rel}(F) \\
&= \textstyle\coprod_{\beta} \circ \coprod_{\kappa_X} \circ \mathsf{Rel}(F) \\
&\Rightarrow \textstyle\coprod_{\beta} \circ \mathsf{Rel}(G) = \mathsf{ctx}_\beta\,.
\end{aligned}$$

This is a natural transformation in Rel_X, which just means that $\mathsf{ctx}_\alpha \leq \mathsf{ctx}_\beta$. $\square$

Proof (of Theorem 7.3). By Lemma 7.1, B satisfies the hypotheses of Theorem 5.1, and hence by Proposition 5.1, ϵ_ω is the algebra induced by the distributive law τ of the companion. This means that $(B_\omega, \epsilon_\omega, \zeta)$ is a τ-bialgebra, and it follows from [7, Corollary 6.8] that $\mathsf{ctx}_{\epsilon_\omega}$ is b_ζ-compatible. Thus $\mathsf{ctx}_{\epsilon_\omega} \leq t_\zeta$.

Let $\alpha\colon FB_\omega \to B_\omega$ be causal. By Corollary 6.1, there is a natural transformation $\hat{\alpha}\colon F \Rightarrow \mathsf{C}_{\bar{B}}$ such that $\alpha = \epsilon_\omega \circ \hat{\alpha}_{B_\omega}$. By Lemma K.1 we obtain $\mathsf{ctx}_\alpha \leq \mathsf{ctx}_{\epsilon_\omega}$, hence $\mathsf{ctx}_\alpha \leq t_\zeta$.

L Proof of Lemma 8.1

Proof. The transformation α is determined by the successor case given in the definition. Naturality is proved in a similar way as in Lemma 5.1, with the relevant diagram in the successor case replaced by:

$$\begin{tikzcd}
FBB_j \arrow[r, "F\langle \mathsf{id}, B_{j+1,j}\rangle"] \arrow[d, "FBB_{j,i}"'] & F(B\times\mathsf{Id})B_j \arrow[r, "\lambda_{B_j}"] \arrow[d, "F(B\times\mathsf{Id})B_{j,i}"] & BF^*B_j \arrow[r, "B\alpha_j^*"] \arrow[d, "BF^*B_{j,i}"] & BB_j \arrow[d, "BB_{j,i}"] \\
FBB_i \arrow[r, "F\langle \mathsf{id}, B_{i+1,i}\rangle"'] & F(B\times\mathsf{Id})B_i \arrow[r, "\lambda_{B_i}"'] & BF^*B_i \arrow[r, "B\alpha_i^*"'] & BB_i
\end{tikzcd}$$

The left square commutes since $B_{i+1,i} \circ BB_{j,i} = B_{i+1,i} \circ B_{j+1,i+1} = B_{j+1,i} = B_{j,i} \circ B_{i+1,i}$ by functoriality and definition of the final sequence. The middle square commutes by naturality. The one on the right commutes, since $B_{j,i}$ is (by assumption in the inductive proof) an algebra morphism, i.e., $B_{j,i} \circ \alpha_j = \alpha_i \circ FB_{j,i}$, and hence $B_{j,i} \circ \alpha_j^* = \alpha_i^* \circ F^*B_{j,i}$ (it holds in general that the $(-)^*$ construction preserves algebra homomorphisms).

Suppose $B_{k+1,k} \colon B_{k+1} \to B_k$ is an isomorphism. Then $B^{-1}_{k+1,k} \colon B_k \to B(B_{k+1})$ is a final B-coalgebra. Consider the following diagram:

$$
\begin{tikzcd}
FB_k \arrow[r, "F\langle B^{-1}_{k+1,k}, \mathsf{id}\rangle"] \arrow[dr, "FB^{-1}_{k+1,k}"'] \arrow[dd, "\alpha_k"'] & F(B \times \mathsf{Id})B_k \arrow[r, "\lambda_{B_k}"] & BF^*B_k \arrow[dd, "B\alpha^*_k"] \\
& FBB_k \arrow[u, "F\langle \mathsf{id}, B_{i+1,i}\rangle"'] \arrow[dr, "\alpha_{k+1}"] & \\
B_k \arrow[rr, "B^{-1}_{k+1,k}"'] & & BB_k
\end{tikzcd}
$$

The big triangle commutes by naturality and the fact that $B_{k+1,k}$ is an isomorphism, the small triangle since $B_{k+1,k}$ is an isomorphism, and the remaining inner shape by definition of α. Hence, α_k is the algebra induced on the final coalgebra by λ. □

M Proof of Theorem 8.1

Proof. By Lemma 8.1, the algebra induced by an abstract GSOS λ is given by α_ω for some $\alpha \colon F\bar{B} \Rightarrow \bar{B}$. By the universal property of the codensity monad $(\mathsf{C}_{\bar{B}}, \epsilon)$, there exists a (unique) natural transformation $\hat{\alpha} \colon F \Rightarrow \mathsf{C}_{\bar{B}}$ such that $\alpha = \epsilon \circ \hat{\alpha}_B$. This means in particular that $\alpha_k = \epsilon_k \circ \hat{\alpha}_{B_k}$. By Proposition 5.1, ϵ_ω is the algebra induced by a distributive law, so α_k is definable by a distributive law over B. □